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A variational approach to the magneto-elastic buckling problem of an arbitrary number of superconducting beams

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Abstract. Based upon a variational principle and the associated theory derived in three preceding papers, an expression for the magneto-elastic buckling value for a system of an arbitrary number of parallel superconducting beams is given. The total current is supposed to be equal both in magnitude and direction for all beams, and the cross-sections are circular. The expression for the buckling value is formulated more explicitly in terms of the so-called buckling amplitudes, the latter following from an algebraic eigenvalue problem. The pertinent matrix is formulated in terms of complex functions, which are replaced by real potentials. The matrix elements are calculated by a numerical method, solving a set of integral equations with regular kernels. Apart from the buckling value(s) the buckling modes are also obtained. Finally, our results are compared with the results of a mathematically less complicated theory, i.e. the method of Biot and Savart.

1. Introduction

In this paper the variational character of the method for the calculation of the magneto-elastic buckling value for superconducting structural systems is shown to full advantage. This variational method is derived in [1] and applied in [2] and [3] to pairs of superconducting beams and rings, respectively. Instead of using the explicit relations for the buckling values, as [2], (1.6) and [3], (2.25), we here start anew with the formulation of a functional $J = J(\mathbf{u}; I_0)$ (taken from [1]). In this, $\mathbf{u}$ is the displacement field (in buckling) and I_0 is the total electric current of the superconducting (slender) structure. This functional J is given by [1], (7.10). Moreover, we consider the relations [1], (7.12), (or [2], (1.7), (1.8)) and [1], (7.15), (or [2], (1.9)) as constraints. Since this paper concerns systems of superconducting beams, we will use the normalized variables as introduced in [2], (3.1). We then can derive from (7.10) (along the same lines as [1], (7.18) is derived) the following expression for J (for the definition of the symbols we refer to [1], [2])

$$
\begin{aligned}
J(\mathbf{u}; I_0) = &- \frac{4\pi^2 ER^2}{\mu_0 I_0^2} \frac{1}{1+\nu} \int_{G^-} \left(\frac{\nu}{1-2\nu} e_{kk} e_{ll} + e_{kl} e_{kl} \right) \mathrm{d}V \\
&+ \int_{\partial G} \Big[\psi(B_j u_{i,j} - B_{i,j} u_j) + B_k B_{k,j} u_i u_j - e_{ijm} B_m A_{j,kl} u_k u_l \\
&+ 2B_k (e_{ijk} u_l - e_{ljk} u_i)(A_{j,m} u_m)_{,l} + \frac{1}{2} B_k B_k (u_{j,j} u_i - u_{i,j} u_j) \Big] N_i \, \mathrm{d}S \\
&- \int_{G^-} T_{jk} u_{i,k} u_{i,j} \, \mathrm{d}V \,, \qquad (1.1)
\end{aligned}
$$

where the intermediate (or rigid-body) fields $\mathbf{B}$ and $\mathbf{A}$ must be determined from (cf. [2], (1.7), (1.8))

$$B_i = e_{ijk}A_{k,j}\,,$$

$$e_{ijk}B_{k,j} = 0\,, \quad (\text{or } A_{i,jj} - A_{j,ij} = 0)\,, \quad \mathbf{x} \in G^+\,; \tag{1.2.1}$$

$$B_i N_i = 0\,, \quad (\text{or } \mathbf{A} = \text{constant})\,, \quad \mathbf{x} \in \partial G\,; \tag{1.2.2}$$

$$\mathbf{B} \to \mathbf{c}(\mathbf{x})\,, \qquad |\mathbf{x}| \to \infty\,; \tag{1.2.3}$$

and the pre-stresses T_{ij} have to satisfy

$$T_{ij,j} = 0\,, \quad \mathbf{x} \in G^-\,; \qquad T_{ij}N_j = -\frac{1}{2}(\mathbf{B}, \mathbf{B})N_i\,, \quad \mathbf{x} \in \partial G\,; \tag{1.3}$$

whereas the perturbed magnetic potential ψ is related to the displacement field $\mathbf{u}$ according to

$$\begin{aligned} &\Delta\psi = 0\,, \quad \mathbf{x} \in G^+\,; \qquad \frac{\partial \psi}{\partial N} = (B_j u_{i,j} - B_{i,j}u_j)N_i\,, \quad \mathbf{x} \in \partial G\,; \\ &\psi \to 0\,, \qquad |\mathbf{x}| \to \infty\,. \end{aligned} \tag{1.4}$$

The constraints for ψ are so severe that, given $\mathbf{u}$, ψ is completely determined by (1.4) (that is why we used the notation $J = J(\mathbf{u}; I_0)$ instead of $J = J(\mathbf{u}, \psi; I_0)$).

We now can propose the following variational approach to the magneto-elastic buckling problem of a superconducting structural system:

the displacement field $\mathbf{u}$ is derived from the variation of J with respect to $\mathbf{u}$ and, then, the buckling value for the current I_0 is obtained by putting J equal to zero (see [1], (2.14)); hence, this means that we have to solve

$$\delta_{\mathbf{u}} J = 0 \quad \text{and} \quad J = 0\,. \tag{1.5}$$

The function $\mathbf{c}(\mathbf{x})$ in the relation (1.2.3) is characteristic of the problem under consideration, but (after the normalization) independent of the current I_0 (see e.g. [2], (3.3)). This relation is made more specific in Section 2, eq. (2.3). Therefore, the current I_0 only turns up in the functional J through the factor $4\pi^2 E R^2/\mu_0 I_0^2$ in the first term of J (see (1.1)), and so the buckling current can indeed be calculated by $(1.5)^2$. The approach to calculate $\mathbf{u}$ from $(1.5)^1$ is different from that in [2] and [3], where $\mathbf{u}$ was chosen a priori (however, based on rather trivial physical arguments).

In the next section we shall apply the method described above to a system of an arbitrary number N of slender superconducting beams, placed parallel to each other in one plane. We shall choose the displacements of the respective beams out of a class of displacement fields representing the bending of a slender beam. The best member of this class is found by application of $(1.5)^1$. In this way an eigenvalue problem for the amplitudes of the buckling displacements of the beams is found. This eigenvalue problem is governed by a symmetric matrix A. The highest eigenvalue of A corresponds to the lowest buckling value of I_0. For the calculation of the matrix A the fields $\mathbf{B}$ and ψ are needed. The main part of this paper is concerned with the calculation of these fields. For $N > 2$ it seems no longer possible to find an analytical solution for $\mathbf{B}$ and ψ (as in [2]) and, therefore, we have to set up a numerical procedure for this calculation. This procedure is presented in Section 4. In Section 5 the

numerical results are given. In the final section some specific results are presented and a comparison with the so-called Biot–Savart-method (cf. [2], [3]) is made.

2. A set of N parallel beams

In [2], Section 4, the authors gave a detailed description of a system of two infinitely long parallel slender beams. For the choice of the coordinate axes $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, we refer to Fig. 1. We restrict ourselves to beams having circular cross-sections, radius R (this is not necessary at this point, since the following analysis analogously holds for cross-sections which show double symmetry; cf. [2], Section 4). The centers of the cross-sections all lie on the $\mathbf{e}_1$-axis at distances $2a$ from each other. The infinitely long beams are periodically supported over length l. We number the N beams with n, $(1 \leq n \leq N)$. The central line of the first beam coincides with the $\mathbf{e}_3$-axis. The regions occupied by the cross-sections in the $\mathbf{e}_1$–$\mathbf{e}_2$-plane are denoted by D_n^-, $(1 \leq n \leq N)$, with boundaries ∂D_n, and the 2-dimensional vacuum space outside the beams is D^+. The position of the center of D_n is $\mathbf{x}_n = 2(n-1)a\mathbf{e}_1$.

In the sequel it is supposed that the total currents, running along the surfaces of the superconducting beams, are all equal both in magnitude (I_0) and in direction. In the undeformed state of the system the currents are in the positive $\mathbf{e}_3$-direction. Analogous to [2], (4.1), the displacement field $\mathbf{u}^{(n)}(\mathbf{x})$, $\mathbf{x} \in D_n^-$, of the n-th beam is expressed in terms of explicit functions of the in-plane variables x and y and the displacement $w_n(z)$ of the central line, according to

$$
\begin{aligned}
u_1^{(n)}(x, y, z) &= w_n(z) + \frac{1}{2}\,\nu[(x - x_n)^2 - y^2]w_n''(z)\,, \\
u_2^{(n)}(x, y, z) &= \nu(x - x_n)yw_n''(z)\,, \\
u_3^{(n)}(x, y, z) &= -(x - x_n)w_n'(z)\,, \quad (x, y) \in D_n^-\,; \\
&\left(x_n = 2(n-1)a\,; \quad ' = \frac{\mathrm{d}}{\mathrm{d}z}\,; \quad 1 \leq n \leq N\right).
\end{aligned}
\tag{2.1}
$$

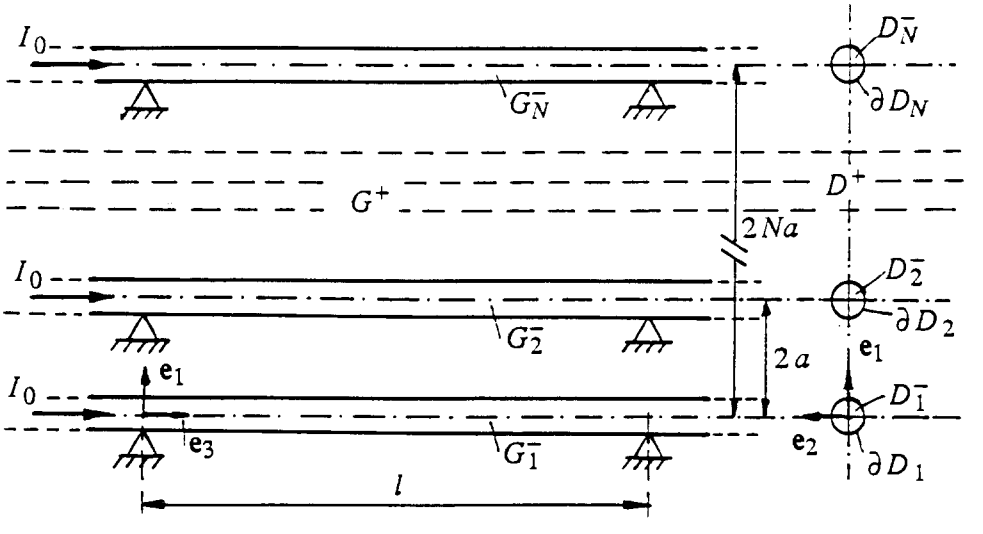

Fig. 1. A set of N parallel beams.

As in [2], (2.5), the problem (1.4) for the perturbed magnetic potential ψ is reduced to a 2-dimensional problem by the separation of variables

$$\psi(x, y, z) = \phi(x, y)w(z)\,, \tag{2.2}$$

(the relationship between $w_n(z)$ and $w(z)$ will be derived further on, see (2.7)). The intermediate (or rigid-body) field $\mathbf{B}$ (subject to the constraints (1.2)) is already purely 2-dimensional, i.e. $\mathbf{B} = \mathbf{B}(x, y)$ and $(\mathbf{B}, \mathbf{e}_3) = 0$. The condition at infinity, (1.2.3), is replaced by the set of conditions (compare with [3], (2.6)) ($\boldsymbol{\tau}$ is the unit tangential vector along ∂D_n)

$$\begin{aligned} &\mathbf{B} \to \mathbf{0}\,, \qquad x^2 + y^2 \to \infty\,, \\ &\int_{\partial D_n} (\mathbf{B}, \boldsymbol{\tau})\,\mathrm{d}s = 2\pi R\,, \quad 1 \leqslant n \leqslant N\,, \end{aligned} \tag{2.3}$$

where the last condition (i.e. Ampère's law in the normalized variables) expresses the relation between the (normalized) rigid-body field $\mathbf{B}$ on the boundary ∂D_n and the total current on the n-th beam.

The constraints (1.2) for the rigid-body field $\mathbf{B} = B_x(x, y)\mathbf{e}_1 + B_y(x, y)\mathbf{e}_2$, can now be written out explicitly, yielding

$$\begin{aligned} &\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0\,, \qquad \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x}\,, \quad (x, y) \in D^+\,; \\ &B_x N_x + B_y N_y = 0\,, \quad (x, y) \in \partial D_n\,; \\ &\int_{\partial D_n} (-B_x N_y + B_y N_x)\,\mathrm{d}s = 2\pi R\,, \quad (1 \leqslant n \leqslant N)\,; \\ &(B_x, B_y) \to 0\,, \qquad x^2 + y^2 \to \infty\,. \end{aligned} \tag{2.4}$$

With respect to (1.3) we only note that, in accordance with the boundary condition $(1.3)^2$ the normalized pre-stresses T_{ij} are of the order of $B^2 = (\mathbf{B}, \mathbf{B})$. The constraints (1.4) for ψ can be evaluated by substitution of (2.1) and (2.2) into them. In doing so we neglect terms of order R^2/l^2. This means in practice, that we maintain in (2.1) only the zeroth order term, i.e.

$$u_1^{(n)} = w_n(z)\,, \qquad u_2^{(n)} = u_3^{(n)} = 0\,. \tag{2.5}$$

The boundary condition $(1.4)^2$ thus becomes

$$\frac{\partial \psi}{\partial N} = \frac{\partial \phi(x, y)}{\partial N}\, w(z) = -\frac{\partial B_x(x, y)}{\partial N}\, w_n(z)\,, \quad (x, y) \in \partial D_n\,. \tag{2.6}$$

Since this relation must be satisfied for arbitrary z, it is necessary that

$$w_n(z) = v_n w(z)\,, \qquad (v_n \in \mathbb{R}\,, \quad 1 \leqslant n \leqslant N)\,. \tag{2.7}$$

We call the numbers v_n the amplitudes of the buckling displacements, and we note that the v_n's are independent of each other. Furthermore, the separation (2.2) is only then consistent with the Laplace equation $(1.4)^1$ if there exists a parameter $\lambda \in \mathbb{R}^+$ such that

$$\Delta\phi(x, y) - \lambda^2\phi(x, y) = 0 \quad \text{and} \quad w''(z) + \lambda^2 w(z) = 0\,. \tag{2.8}$$

The parameter λ is related to l through the support conditions of the beams (which are supposed to be the same for all beams). For simply supported beams λ equals π/l.

In this way the following constraint relations for $\phi(x, y)$ are obtained from (1.4)

$$\begin{aligned} &\Delta\phi = \lambda^2\phi\,, \quad (x, y) \in D^+\,; \\ &\frac{\partial\phi}{\partial N} = -v_n \frac{\partial B_x}{\partial N}\,, \quad (x, y) \in \partial D_n\,, \quad (1 \leqslant n \leqslant N)\,; \\ &\phi \to 0\,, \qquad x^2 + y^2 \to \infty\,. \end{aligned} \tag{2.9}$$

The amplitudes v_n of the central line displacements and the buckling value for I_0 are still unknown and are to be solved from the variation and zeroness of the functional J, i.e.

$$\frac{\partial J}{\partial v_n} = 0 \quad (1 \leqslant n \leqslant N) \quad \text{and} \quad J = 0\,. \tag{2.10}$$

We proceed with the evaluation of the expression for J according to (1.1) for the displacement field (2.1). Firstly, we note that in the formula (1.1) for the functional J the regions G^+, G^- and the boundary ∂G are to be restricted to the truncations $D^+ \times [0, p]$, $D^- \times [0, p]$ and $\partial D \times [0, p]$, respectively, where D^- and ∂D are the unions of the regions D_n^- and the boundaries ∂D_n, respectively. This is based upon the assumption that the fields are periodic in the z- or $\mathbf{e}_3$-direction with period p (see [2], section 2, for more details).

The right-hand side of (1.1) contains three integrals. The first one, representing the elastic energy, yields in the usual way the classical bending energy for a slender beam (see [2], (2.2)). Since we neglect terms of $O(R^2/l^2)$ (or $O(\lambda^2 R^2)$, as λ is proportional to l^{-1}) we may use in the evaluation of the second integral the reduced form (2.5) for the displacement field. Moreover, we use (2.2), $(2.4)^{1,2}$, (2.7) and $(2.8)^2$, and we introduce the set of functions ϕ_m, $(1 \leqslant m \leqslant N)$, by

$$\phi(x, y) = \sum_{m=1}^{N} v_m \phi_m(x, y)\,. \tag{2.11}$$

Then (2.9) implies that each ϕ_m is independent of the amplitudes $v_1, v_2, \ldots v_N$, and has to satisfy

$$\begin{aligned} &\Delta\phi_m = \lambda^2\phi_m\,, \quad (x, y) \in D^+\,; \\ &\frac{\partial\phi_m}{\partial N} = -\frac{\partial B_x}{\partial N}\,, \quad (x, y) \in \partial D_m\,, \qquad \frac{\partial\phi_m}{\partial N} = 0\,, \quad (x, y) \in \partial D \backslash \partial D_m\,; \\ &\phi_m \to 0\,, \quad x^2 + y^2 \to \infty\,, \end{aligned} \tag{2.12}$$

for each $m \in [1, N]$.

Finally, we note that (as T_{ij} is of the order B^2) the third integral gives a contribution that is of $O(R^2/l^2)$ and, hence, negligible (just as was found in [2] and [3]). All this yields, apart from a factor

$$\int_0^p w^2(z)\,\mathrm{d}z$$

(which might be normalized to unity), finally the expression for the functional J, i.e.

$$J = J(\mathbf{v}; I_0) = (A\mathbf{v}, \mathbf{v}) - \kappa(\mathbf{v}, \mathbf{v}) , \tag{2.13}$$

which is exact up to $O(\lambda^2 R^2(\mathbf{v}, \mathbf{v}))$. Here $\mathbf{v}$ is a N-vector, representing the buckling amplitudes, which possesses the following column representation with regard to the orthonormal, positively orientated base $\{\mathbf{E}_1, \ldots, \mathbf{E}_N\}$ of $\mathbb{R}_N$,

$$\mathbf{v} = [v_1, v_2, \ldots, v_N]^T ; \tag{2.14}$$

κ is a positive scalar, which represents the entrance into the functional of the current I_0,

$$\kappa = \frac{4\pi^2 E I_y \lambda^4 R^2}{\mu_0 I_0^2} , \qquad I_y = \int_{D_1^-} x^2 \, \mathrm{d}S = \frac{1}{4} \pi R^4 , \tag{2.15}$$

and A is a linear transformation from $\mathbb{R}_N \to \mathbb{R}_N$, having the following matrix with regard to the base $\{\mathbf{E}_1, \ldots, \mathbf{E}_N\}$,

$$\begin{aligned} A_{mn} &= -\int_{\partial D_n} \phi_m \frac{\partial B_x}{\partial N} \, \mathrm{d}s , \quad 1 \leqslant m, n \leqslant N , \quad m \neq n ; \\ A_{nn} &= -\int_{\partial D_n} (\phi_n + B_x) \frac{\partial B_x}{\partial N} \, \mathrm{d}s , \quad 1 \leqslant n \leqslant N . \end{aligned} \tag{2.16}$$

Due to the Helmholtz problem (2.12) and Green's second identity we derive from the matrix representation formulas (2.16) the property

$$A_{mn} = \int_{\partial D} \phi_m \frac{\partial \phi_n}{\partial N} \, \mathrm{d}s = \int_{\partial D} \phi_n \frac{\partial \phi_m}{\partial N} \, \mathrm{d}s = A_{nm} , \quad n \neq m . \tag{2.17}$$

Hence, the linear transformation A is symmetric and elaboration of (2.10) yields

$$A\mathbf{v} = \kappa \mathbf{v} , \quad \mathbf{v} \neq \mathbf{0} , \quad \kappa > 0 ; \qquad \kappa = \frac{(A\mathbf{v}, \mathbf{v})}{(\mathbf{v}, \mathbf{v})} . \tag{2.18}$$

The set (2.18) implies that the lowest buckling value for the current I_0 corresponds to the highest positive eigenvalue κ of the matrix A. This matrix still depends on the parameter λ by means of the functions ϕ_m (cf. $(2.12)^1$). In the next section we shall prove that for slender beams the influence of the ratio R/l on the eigenvalue for κ is negligible.

3. Complex formulation

In this section we shall use a great deal of the complex manipulations, which were already applied to the buckling problems for one single beam and for a set of two parallel beams in [2]. Therefore, we shall recapitulate only those notations and methods, which are indispensable to the understanding of the complete procedure. We introduce a small parameter δ $(0 < \delta \ll 1)$, the normalized complex coordinate z and the complex function F in the same way as in [2], (2.7), (3.7), (3.25), i.e.

$$\delta = \lambda R\,, \qquad z = (x + \mathrm{i}y)/R\,,$$
$$F = B_x - \mathrm{i}B_y\,, \quad z \in S^+ \cup C\,, \tag{3.1}$$

where S^+ and C stand for the region and curves in the complex z-plane corresponding to D^+ and ∂D, respectively. Moreover, we denote the z-transformations of D_n^- and ∂D_n by S_n^- and C_n, respectively.

Analogous to [2], (3.26), (4.2), (4.4) the relations for the rigid-body state (see (2.4)) can be transformed into (for the definition of the complex line element $\mathrm{d}z$ see [2], (3.22))

$$F \text{ analytical}\,, \quad z \in S^+\,,$$
$$F\,\mathrm{d}z \in \mathbb{R}\,, \quad z \in C\,,$$
$$F \to 0\,, \quad |z| \to \infty\,, \tag{3.2}$$
$$\int_{C_n} F\,\mathrm{d}z = 2\pi\,, \quad 1 \leqslant n \leqslant N\,.$$

The introduction of the real-valued functions (compare with [2], (3.28), (4.5) and note the difference between the definition of f_m used here and the one according to [2], (4.5))

$$\begin{aligned} f_m(z, \bar{z}) &= \phi_m\,, && 1 \leqslant |z - z_n| \leqslant a/R\,, \quad n \neq m\,, \\ &= (\phi_m + B_x)\,, && 1 \leqslant |z - z_m| \leqslant a/R\,, \end{aligned} \tag{3.3}$$

$(1 \leqslant m, n \leqslant N)$ enables us to write (2.16) as (for the definition of the complex derivative $\partial/\partial z$, see [2], (3.24))

$$A_{mn} = -2\int_{C_n} f_m\,\mathrm{Im}\,\frac{\partial B_x}{\partial z}\,\mathrm{d}z = -\mathrm{Im}\int_{C_n} f_m\,\frac{\mathrm{d}F}{\mathrm{d}z}\,\mathrm{d}z\,, \tag{3.4}$$

and $(2.12)^2$ as

$$\frac{\partial f_m}{\partial N} = 0\,, \quad z \in C\,, \quad 1 \leqslant m, n \leqslant N\,. \tag{3.5}$$

What we are looking for are the numerical values of the coefficients A_{mn} according to (3.4) and, hence, it is evident that our special interest is in the boundary values of the functions f_m. For the calculation of these values an integral equation is constructed. Since the construction runs along the lines of the methods presented in [2], (3.31)–(3.46) and (4.7)–(4.15), we do not enter into further details here, but only state the main results. Also, we use the convention that any $O(\delta^2 \log^k \delta)$-term is referred to as an $O(\delta^2)$-term.

The functions f_m are asymptotically approximated by the δ-independent functions g_m, according to

$$f_m(z) = g_m(z)(1 + O(\delta^2))\,, \quad z \in C\,, \quad 1 \leqslant m \leqslant N\,, \tag{3.6}$$

where g_m satisfies (compare with [2], (4.10.2))

$$\frac{1}{2}\,g_m(z_0) + \mathrm{Re}\left\{\frac{1}{2\pi\mathrm{i}}\int_C \frac{g_m(z)}{z - z_0}\,\mathrm{d}z\right\} = R(z_0)\,, \tag{3.7}$$

with

$$R(z_0) = \mathrm{Re}\left\{ \frac{1}{2\pi \mathrm{i}} \int_{C_m} \frac{F(z)}{z - z_0} \, \mathrm{d}z \right\}, \quad z_0 \in C \backslash C_m , \tag{3.8.1}$$

and

$$R(z_0) = \mathrm{Re}\left\{ \frac{1}{2\pi \mathrm{i}} \int_{C \backslash C_m} \frac{F(z)}{z - z_0} \, \mathrm{d}z \right\}, \quad z_0 \in C_m , \tag{3.8.2}$$

Cauchy's theorem for analytical functions states that

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \int_C \frac{F(z)}{z - z_0} \, \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_{C_m} \frac{F(z)}{z - z_0} \, \mathrm{d}z + \frac{1}{2\pi \mathrm{i}} \int_{C \backslash C_m} \frac{F(z)}{z - z_0} \, \mathrm{d}z = 0 , \qquad & z_0 \in S^- , \\ = -F(z_0) , \quad & z_0 \in S^+ . \end{aligned} \tag{3.9}$$

Introduction of the N analytical functions (so-called Cauchy-integrals)

$$\Phi_m(z_0) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{g_m(z)}{z - z_0} \, \mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \int_{C_m} \frac{F(z)}{z - z_0} \, \mathrm{d}z , \quad z_0 \in \mathbb{C} \backslash C , \tag{3.10}$$

and use of (3.9) in (3.7)–(3.8) leads us to the following set of Riemann–Hilbert problems

$$\mathrm{Re}\, \Phi_m^-(z_0) = 0 , \quad z_0 \in C , \tag{3.11.1}$$

and

$$\begin{aligned} \mathrm{Im}[\Phi_m^-(z_0) - \Phi_m^+(z_0)] = -\mathrm{Im}\, F(z_0) , \quad & z_0 \in C_m , \\ = 0 , \qquad & z_0 \in C \backslash C_m . \end{aligned} \tag{3.11.2}$$

Furthermore, the functions g_m are related to the Cauchy-integrals Φ_m,

$$\begin{aligned} g_m(z_0) = \Phi_m^-(z_0) - \Phi_m^+(z_0) + F(z_0) , \quad & z_0 \in C_m , \\ = \Phi_m^-(z_0) - \Phi_m^+(z_0) , \qquad & z_0 \in C \backslash C_m . \end{aligned} \tag{3.12}$$

Since Φ_m is analytical in S^- it follows from (3.11.1) that Φ_m^- equals an imaginary constant, i.e.

$$\Phi_m^-(z) = \mathrm{i} c_{mn} , \quad z \in S_n^- , \quad c_{mn} \in \mathbb{R} . \tag{3.13}$$

Substitution of (3.6), (3.12) and (3.13) into the expression for A_{mn} according to (3.4) yields, under the neglect of $O(\delta^2)$-terms,

$$\begin{aligned} A_{mn} &= -\mathrm{Im} \int_{C_n} g_m \frac{\mathrm{d}F}{\mathrm{d}z} \, \mathrm{d}z = \mathrm{Im} \int_{C_n} F \frac{\mathrm{d}g_m}{\mathrm{d}z} \, \mathrm{d}z \\ &= \mathrm{Im} \int_{C_n} F \frac{\mathrm{d}}{\mathrm{d}z} (\mathrm{i} c_{mn} - \Phi_m^+ + \delta_{mn} F) \, \mathrm{d}z = -\mathrm{Im} \int_{C_n} F \frac{\mathrm{d}\Phi_m^+}{\mathrm{d}z} \, \mathrm{d}z . \end{aligned} \tag{3.14}$$

Using the short-hand notation

$$F_m(z) = \frac{\mathrm{d}\Phi_m^+}{\mathrm{d}z}, \quad z \in S^+ \cup C, \tag{3.15}$$

we arrive at the ultimate mathematical formulation for the determination of the buckling current I_0:

Calculate the matrix A from

$$A_{mn} = -\mathrm{Im} \int_{C_n} F F_m \, \mathrm{d}z, \quad 1 \leqslant m, n \leqslant N, \tag{3.16}$$

where the functions $F(z)$ and $F_m(z)$ satisfy

$$\begin{aligned}
&F, F_m \text{ analytical}, \quad z \in S^+, \\
&F, F_m \to 0, \quad |z| \to \infty, \\
&\int_{C_n} F \, \mathrm{d}z = 2\pi, \qquad \int_{C_n} F_m \, \mathrm{d}z = 0, \\
&\mathrm{Im}(F \, \mathrm{d}z) = 0, \quad z \in C, \\
&\mathrm{Im}(F_m \, \mathrm{d}z) = \mathrm{Im}\left(\frac{\mathrm{d}F}{\mathrm{d}z} \, \mathrm{d}z\right), \quad z \in C_m, \\
&\qquad\qquad = 0, \qquad\qquad\quad z \in C \backslash C_m;
\end{aligned} \tag{3.17}$$

and, then, the amplitude-vector $\mathbf{v}$ *and the buckling current I_0 are obtained from the eigenvalue problem*

$$A\mathbf{v} = \kappa \mathbf{v}, \quad \mathbf{v} \neq \mathbf{0}, \quad \kappa > 0, \tag{3.18}$$

and the relation

$$I_0 = 2\pi\delta^2 \left(\frac{EI_y}{\mu_0 \kappa R^2}\right)^{1/2}. \tag{3.19}$$

On account of the fact that, within our approximation, the matrix A is independent of the parameter δ, it is evident that the buckling current I_0 is proportional to δ^2. Moreover, we note that (3.17) directly implies that

$$\sum_{m=1}^{N} F_m = \frac{\mathrm{d}F}{\mathrm{d}z}, \quad z \in S^+ \cup C, \tag{3.20}$$

and as a consequence, the column-sums of the matrix A are equal to zero. Use of this property in (3.18) shows us that

$$\sum_{m=1}^{N} v_m = 0. \tag{3.21}$$

In other words, the amplitudes of the central line displacements always cancel each other.

4. Numerical procedure for the calculation of the matrix A

In [2], for the case of two circular rods, the region S^+ was transformed into a ringshaped region by conformal mapping and the resulting problem was solved by complex analysis. For the case $N>2$ such an analytical treatment is impossible and, therefore, we search for a numerical solution procedure for the eigenvalue problem (3.18). This, more specifically, amounts to a numerical calculation of the elements A_{mn} of the matrix A, according to (3.16).

The first step is to reformulate the problem (3.16)–(3.20) in real terms, by introduction of the real functions $\omega = \omega(x, y)$ and $\omega_m = \omega_m(x, y)$ through

$$F = -\frac{\partial \omega}{\partial y} - \mathrm{i}\,\frac{\partial \omega}{\partial x}\,; \qquad F_m = -\frac{\partial \omega_m}{\partial y} - \mathrm{i}\,\frac{\partial \omega_m}{\partial x}\,, \tag{4.1}$$

for $1 \leq m \leq N$ and $\mathbf{x} = (x, y) \in S^+ \cup C$.

The problem then transforms into (with $\mathrm{d}z = \mathrm{i}N\,\mathrm{d}s = (\mathrm{i}N_x - N_y)\,\mathrm{d}s$, and $\partial\omega/\partial s = 0$ (see (4.3.3))):

Find the positive eigenvalues κ of the matrix A with elements

$$A_{mn} = \int_{C_n} \frac{\partial \omega_m}{\partial x}\,\frac{\partial \omega}{\partial N}\,\mathrm{d}s\,, \quad 1 \leq m, n \leq N\,; \tag{4.2}$$

where ω and ω_m satisfy

$$\Delta\omega = 0\,, \qquad \Delta\omega_m = 0\,, \quad \mathbf{x} \in S^+\,; \tag{4.3.1}$$

$$\nabla\omega \to \mathbf{0}\,, \qquad \nabla\omega_m \to \mathbf{0}\,, \quad |\mathbf{x}| \to \infty\,; \tag{4.3.2}$$

$$\frac{\partial \omega}{\partial s} = 0\,, \quad \mathbf{x} \in C\,; \qquad \frac{\partial \omega_m}{\partial s} = \delta_{mn}\,\frac{\partial}{\partial s}\left(N_x\,\frac{\partial \omega}{\partial N}\right), \quad \mathbf{x} \in C_n\,; \tag{4.3.3}$$

$$\int_{C_n} \frac{\partial \omega}{\partial N}\,\mathrm{d}s = 2\pi\,, \qquad \int_{C_n} \frac{\partial \omega_m}{\partial N}\,\mathrm{d}s = 0\,, \tag{4.3.4}$$

for $1 \leq m, n \leq N$.

With (4.3.1) and (4.3.4) the conditions at infinity (4.3.2) can be made more explicit, yielding

$$\omega = N \log|\mathbf{x}| + O(1)\,, \qquad \omega_m = O(1)\,, \quad |\mathbf{x}| \to \infty\,. \tag{4.4}$$

If wished for, the $O(1)$-terms (constants) in (4.4) can be made zero, i.e. replaced by $o(1)$-terms, because the potentials ω and ω_m are only relevant up to a constant term. Moreover, the boundary conditions (4.3.3) can be integrated along each separate boundary C_n, giving

$$\omega = \alpha_n\,, \qquad \omega_m = \delta_{mn} N_x\,\frac{\partial \omega}{\partial N} + \beta_{nm}\,, \quad \mathbf{x} \in C_n\,, \tag{4.5}$$

where α_n and β_{nm} are constant factors, which shall be determined further on from (4.3.4).

In the second step the functions ω and ω_m are split up in a set of harmonic functions (in

$S^+ \cup C$), which are bounded at infinity and known on the boundary C, according to ($\mathbf{x}_k = 2(k-1)a/R\,\mathbf{e}_1$, the center of the k-th cross-section)

$$\omega = \sum_{k=1}^{N} \log|\mathbf{x} - \mathbf{x}_k| + \psi + \sum_{k=1}^{N} \alpha_k u_k \,,$$
$$\omega_m = \psi_m + \sum_{k=1}^{N} \beta_{km} u_k \,. \tag{4.6}$$

The first term of (4.6) is chosen in such a way that the first condition of (4.3.4) is satisfied. The functions ψ and ψ_m have to satisfy the boundary conditions (4.5) with $\alpha_n = \beta_{nm} = 0$, as the remaining part of these boundary conditions are fulfilled by the parts with u_k. All the unknown functions (i.e. ψ, ψ_m and u_k) can be found from an exterior Dirichlet problem, which in general form reads ($V = V(x, y)$)

$$\Delta V = 0 \,, \quad \mathbf{x} \in S^+ \,,$$
$$V = O(1) \,, \quad |\mathbf{x}| \to \infty \,, \tag{4.7}$$
$$V = f \,, \quad \mathbf{x} \in C \,,$$

where f is a given function of $\mathbf{x}$ on the boundary C of the exterior region S^+. In (4.7) we have to read for V successively ψ, ψ_m and u_k. The associated boundary functions f are given by:

$$\text{for } V = \psi \,, \qquad f(\mathbf{x}) = -\sum_{k=1}^{N} \log|\mathbf{x} - \mathbf{x}_k| \,, \tag{4.8.1}$$

$$\text{for } V = \psi_m \,, \qquad f(\mathbf{x}) = \begin{cases} 0 \,, & \mathbf{x} \in C \backslash C_m \,, \\ N_x \dfrac{\partial \omega}{\partial N} \,, & \mathbf{x} \in C_m \,, \end{cases} \tag{4.8.2}$$

$$\text{for } V = u_k \,, \qquad f(\mathbf{x}) = \begin{cases} 0 \,, & \mathbf{x} \in C \backslash C_k \,, \\ 1 \,, & \mathbf{x} \in C_k \,. \end{cases} \tag{4.8.3}$$

The coefficients α_k and β_{km} are still to be determined from (4.3.4). This results in the following relations (for $1 \leqslant m, n \leqslant N$)

$$\sum_{k=1}^{N} \alpha_k \int_{C_n} \frac{\partial u_k}{\partial N}\, \mathrm{d}s = -\int_{C_n} \frac{\partial \psi}{\partial N}\, \mathrm{d}s \,,$$
$$\sum_{k=1}^{N} \beta_{km} \int_{C_n} \frac{\partial u_k}{\partial N}\, \mathrm{d}s = -\int_{C_n} \frac{\partial \psi_m}{\partial N}\, \mathrm{d}s \,. \tag{4.9}$$

It should be noted that the N relations of the set $(4.9)^1$ and the $N \times N$ relations of $(4.9)^2$ are linearly dependent because (for each $m, k \in [1, N]$)

$$\int_C \frac{\partial \psi}{\partial N}\, \mathrm{d}s = \int_C \frac{\partial \psi_m}{\partial N}\, \mathrm{d}s = \int_C \frac{\partial u_k}{\partial N}\, \mathrm{d}s = 0 \,, \tag{4.10}$$

due to the fact that ψ, ψ_m and u_k are harmonic in S^+ and bounded at infinity. Therefore, in both of the sets of (4.9) one relation has to be dropped. This can be replaced by the following relations at infinity

$$\begin{aligned} &\sum_{k=1}^{N} \alpha_k u_k + \psi = 0\,, \quad |\mathbf{x}| \to \infty\,, \\ &\sum_{k=1}^{N} \beta_{km} u_k + \psi_m = 0\,, \quad (1 \leqslant m \leqslant N)\,, \quad |\mathbf{x}| \to \infty\,. \end{aligned} \tag{4.11}$$

For the derivation of these relations it is necessary to replace in (4.4) the $O(1)$-symbols by $o(1)$-symbols.

With the use of (4.3.3) the expression (4.2) for A_{mn} can be rewritten in the form

$$A_{mn} = \int_{C_n} \left[N_x \frac{\partial \omega_m}{\partial N} - \delta_{mn} N_y \frac{\partial}{\partial s} \left(N_x \frac{\partial \omega}{\partial N} \right) \right] \frac{\partial \omega}{\partial N}\, \mathrm{d}s\,. \tag{4.12}$$

For the calculation of these integrals we first have to solve the basic problems (4.7)–(4.8). However, from (4.12) we see that, practically, we are only interested in the values of the normal derivatives along the boundaries, i.e. $\partial V/\partial N$ for $\mathbf{x} \in C_n$, $1 \leqslant n \leqslant N$.

The further procedure could be based on the use of layer potentials (cf. [4], [5]). However, introduction of a simple layer potential for the function V leads us to a situation in which it is difficult to determine the limit of V at infinity, and, moreover, the problem now involves a Fredholm integral equation of the first kind (weakly singular), i.e. an ill-posed problem for the density of the potential. On the other hand, by introducing a double layer potential we arrive at a Fredholm integral equation of the second kind, which in general is singular.

To avoid these complications, we separate from V particular logarithmic solutions of the Laplace equation. The remaining part of V can then be expressed in double layer potentials, the densities of which satisfy ordinary integral equations. This separation is of the following form

$$V(\mathbf{x}) = V_1(\mathbf{x}) + V_2(\mathbf{x})\,, \quad \mathbf{x} \in S^+ \cup C\,, \tag{4.13}$$

where firstly

$$V_1(\mathbf{x}) = -\frac{1}{2\pi} \int_C \mu(\mathbf{y}) \frac{\partial}{\partial N_y} \log|\mathbf{x} - \mathbf{y}|\, \mathrm{d}s_y\,, \quad \mathbf{x} \in S^+ \cup S^-\,, \tag{4.14}$$

with $\mu(\mathbf{x})$ satisfying

$$\frac{1}{2}\, \mu(\mathbf{x}) - \frac{1}{2\pi} \int_{C \setminus C_n} \mu(\mathbf{y}) \frac{\partial}{\partial N_y} \log|\mathbf{x} - \mathbf{y}|\, \mathrm{d}s_y = f(\mathbf{x})\,, \quad \mathbf{x} \in C_n\,, \tag{4.15}$$

or in short-hand notation

$$L^+\{\mu(\mathbf{x})\} = f(\mathbf{x})\,, \quad \mathbf{x} \in C_n\,, \quad 1 \leqslant n \leqslant N\,. \tag{4.16}$$

Secondly

$$V_2(\mathbf{x}) = c_0 + \sum_{l=1}^{N} c_l \left[\frac{1}{2\pi} \log|\mathbf{x} - \mathbf{x}_l| - V^l(\mathbf{x}) \right], \tag{4.17}$$

for $\mathbf{x} \in S^+ \cup (S^- \backslash \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\})$, where

$$\sum_{l=1}^{N} c_l = 0, \tag{4.18}$$

$$V^l(\mathbf{x}) = -\frac{1}{2\pi} \int_C \mu^l(\mathbf{y}) \frac{\partial}{\partial N_y} \log|\mathbf{x} - \mathbf{y}| \, ds_y, \quad \mathbf{x} \in S^+ \cup S^-, \tag{4.19}$$

while $\mu^l(\mathbf{x})$ has to satisfy

$$L^+\{\mu^l(\mathbf{x})\} = \frac{1}{2\pi} \log|\mathbf{x} - \mathbf{x}_l|, \quad \mathbf{x} \in C_n. \tag{4.20}$$

Evidently

$$\Delta V_1 = 0, \quad \mathbf{x} \in S^+ \cup S^-, \tag{4.21}$$

$$\Delta V_2 = 0, \quad \mathbf{x} \in S^+; \qquad \Delta V_2 = c_l \delta_D(\mathbf{x} - \mathbf{x}_l), \quad \mathbf{x} \in S_l^-, \tag{4.22}$$

(δ_D is Dirac's delta function) and

$$V_1 \to 0, \qquad V_2 \to c_0 = O(1), \quad |\mathbf{x}| \to \infty, \tag{4.23}$$

where the latter is a consequence of (4.18). From (4.23) together with (4.13) it follows that

$$c_0 = V_\infty = \lim_{|\mathbf{x}| \to \infty} V(\mathbf{x}). \tag{4.24}$$

As we shall show further on, the numbers $c_0, c_1, \ldots, c_N$ can be chosen in such a way that $V = f$ on C. Note that the integral equations (4.15) (or (4.16)) and (4.20) possess indeed regular kernels. Moreover, the normal derivatives of the double layer potentials V_1 and V^l are continuous across the boundaries C_n (see [6], p. 170), so (since $\Delta V_1 = 0$ and $\Delta V^l = 0$, $\mathbf{x} \in S_n^-$)

$$\int_{C_n} \frac{\partial V_1}{\partial N} \, ds = 0, \qquad \int_{C_n} \frac{\partial V^l}{\partial N} \, ds = 0, \quad 1 \leqslant l, n \leqslant N, \tag{4.25}$$

and then (from (4.17))

$$c_n = \int_{C_n} \frac{\partial V_2}{\partial N} \, ds = \int_{C_n} \frac{\partial V}{\partial N} \, ds, \quad 1 \leqslant n \leqslant N. \tag{4.26}$$

Taking in (4.14) for $V_1(\mathbf{x})$ the exterior limit for $\mathbf{x} \to C_n$, denoted by $V_1^+(\mathbf{x})$, we arrive at (cf. [4], p. 382; $\fint$ stands for the principal value)

$$\begin{aligned} V_1^+(\mathbf{x}) &= \frac{1}{2} \mu(\mathbf{x}) - \frac{1}{2\pi} \fint_C \mu(\mathbf{y}) \frac{\partial}{\partial N_y} \log|\mathbf{x} - \mathbf{y}| ds_y \\ &= f(\mathbf{x}) - \frac{1}{2\pi} \fint_{C_n} \mu(\mathbf{y}) \frac{\partial}{\partial N_y} \log|\mathbf{x} - \mathbf{y}| \, ds_y, \quad \mathbf{x} \in C_n, \end{aligned} \tag{4.27}$$

where the last step follows immediately from (4.15). Writing for $\mathbf{y}$ and for $\mathbf{x} \in C_n$

$$\mathbf{y} = (x_n + r\cos\phi)\mathbf{e}_1 + r\sin\phi\,\mathbf{e}_2 ,$$

and

$$\mathbf{x} = (x_n + \cos\theta)\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2 ,$$

respectively, we find for $\mathbf{y} \in C_n$ ($\mathbf{N}_y = \cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2$)

$$\frac{\partial}{\partial N_y}\log|\mathbf{x}-\mathbf{y}| = \left[\frac{\partial}{\partial r}\log|\mathbf{x}-\mathbf{y}|\right]_{r=1} = \left[\frac{(-\mathbf{x}+\mathbf{y},\mathbf{N}_y)}{|\mathbf{x}-\mathbf{y}|^2}\right]_{r=1}$$
$$= \frac{1-\cos(\theta-\phi)}{2(1-\cos(\theta-\phi))} = \frac{1}{2} . \tag{4.28}$$

With (4.28) the integral on the right-hand side of (4.27) can be evaluated to (for $\mathbf{x} \in C_n$)

$$\frac{1}{2\pi}\int_{C_n} \mu(\mathbf{y})\frac{\partial}{\partial N_y}\log|\mathbf{x}-\mathbf{y}|\,\mathrm{d}s_y = \frac{1}{2}\,\bar{\mu}_n , \tag{4.29}$$

where $\bar{\mu}_n$ stands for the mean value of μ on C_n, i.e.

$$\bar{\mu}_n = \frac{1}{2\pi}\int_{C_n} \mu\,\mathrm{d}s . \tag{4.30}$$

As a consequence of (4.29), (4.27) reduces to

$$V_1^+(\mathbf{x}) = f(\mathbf{x}) - \frac{1}{2}\,\bar{\mu}_n , \quad \mathbf{x} \in C_n . \tag{4.31}$$

In a similar way one deduces

$$V_2^+(\mathbf{x}) = c_0 + \frac{1}{2}\sum_{m=1}^{N} c_m \bar{\mu}_n^m , \quad \mathbf{x} \in C_n , \tag{4.32}$$

where $\bar{\mu}_n^m$ is the mean value of the density μ^m on C_n. The boundary condition

$$V(\mathbf{x}) = V_1^+(\mathbf{x}) + V_2^+(\mathbf{x}) = f(\mathbf{x}) ,$$

now yields

$$\frac{1}{2}\sum_{m=1}^{N} c_m \bar{\mu}_n^m + c_0 = \frac{1}{2}\,\bar{\mu}_n , \quad 1 \leq n \leq N . \tag{4.33}$$

This set, together with the relation (4.18), which is the necessary condition for the boundedness of $V_2(\mathbf{x})$ at infinity, constitute the basic set for the calculation of $c_0, c_1, \ldots, c_N$ (after μ and μ^l are known). We can write this total set in a more concise notation by introducing the N-column vectors $\mathbf{a}$ and $\mathbf{e}$ and the $(N \times N)$-matrix B by

$$a_n = \frac{1}{2}\,\bar{\mu}_n , \qquad e_n = 1 , \qquad B_{mn} = \frac{1}{2}\,\bar{\mu}_n^m , \tag{4.34}$$

for $1 \leqslant m, n \leqslant N$. Then, the above mentioned set can be written as

$$\begin{bmatrix} B & \mathbf{e} \\ \mathbf{e}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ c_0 \end{bmatrix} = \begin{bmatrix} \mathbf{a} \\ 0 \end{bmatrix}. \tag{4.35}$$

In this system of linear equations the vector $(\mathbf{c}^T, c_0)^T$ represents the unknown variables. The vector $\mathbf{e}$ is a fixed one, whereas the matrix B and the vector $\mathbf{a}$ are known once the ordinary integral equations (4.16) and (4.20) are solved (recall that this must be done for all f's out of the three distinct sets presented in (4.8)). Note also that for the solution of (4.35) we do not need to calculate the functions $V(\mathbf{x})$ or $V_1(\mathbf{x})$ and $V_2(\mathbf{x})$; these are only auxiliary functions. As a matter of fact we are only interested in the values of the normal derivative of V at the boundaries C_n. For this purpose we consider the function

$$V_3(\mathbf{x}) = V(\mathbf{x}) - \frac{1}{2\pi} \sum_{m=1}^{N} c_m \log|\mathbf{x} - \mathbf{x}_m| , \tag{4.36}$$

hence,

$$V_3(\mathbf{x}) = V_1(\mathbf{x}) + c_0 - \sum_{m=1}^{N} c_m V^m . \tag{4.37}$$

From the foregoing analysis it then follows that $V_3(\mathbf{x})$ is harmonic in S^+, bounded at infinity and such that (from (4.25))

$$\int_{C_n} \frac{\partial V_3}{\partial N} \, ds = 0 , \quad 1 \leqslant n \leqslant N . \tag{4.38}$$

These features guarantee the existence of a harmonic function $W(\mathbf{x})$, $\mathbf{x} \in S^+ \cup C$, the conjugate function of V_3, such that

$$\begin{aligned} &\Delta W = 0 , \quad \mathbf{x} \in S^+ , \\ &W = O(1) , \quad |\mathbf{x}| \to \infty , \\ &\frac{\partial W}{\partial N} = -\frac{\partial V_3}{\partial s} = \frac{\partial f}{\partial s} - \frac{1}{2\pi} \frac{\partial}{\partial s} \sum_{m=1}^{N} c_m \log|\mathbf{x} - \mathbf{x}_m| , \quad \mathbf{x} \in C , \end{aligned} \tag{4.39}$$

since $V = f$, for $\mathbf{x} \in C$.

The above problem for W is, apart from an irrelevant constant, uniquely solved by writing W as a simple layer potential, the density of which satisfies an ordinary integral equation with regular kernel. Thus (cf. [4])

$$W(\mathbf{x}) = -\frac{1}{2\pi} \int_C \nu(\mathbf{y}) \log|\mathbf{x} - \mathbf{y}| \, ds_y , \tag{4.40}$$

with ν following from

$$-\frac{1}{2} \nu(\mathbf{x}) - \frac{1}{2\pi} \int_{C \setminus C_n} \nu(\mathbf{y}) \frac{\partial}{\partial N_x} \log|\mathbf{x} - \mathbf{y}| \, ds_y = -\frac{\partial V_3}{\partial s} , \quad \mathbf{x} \in C_n , \tag{4.41}$$

with $\partial V_3 / \partial s$ as given by (4.39)^3. Since W is the conjugate of V_3, the normal derivative of V_3

on C equals the tangential derivative of W along C, so

$$\frac{\partial V}{\partial N} = \frac{\partial W}{\partial s} + \frac{1}{2\pi}\frac{\partial}{\partial N}\sum_{m=1}^{N} c_m \log|\mathbf{x}-\mathbf{x}_m| \,. \tag{4.42}$$

According to (4.40)

$$\begin{aligned}\frac{\partial W}{\partial s} &= -\frac{1}{2\pi}\int_C \nu(\mathbf{y})\frac{(\mathbf{x}-\mathbf{y},\mathbf{s}_x)}{|\mathbf{x}-\mathbf{y}|^2}\,ds_y \\ &= -\frac{1}{2\pi}\int_{C\setminus C_n}\nu(y)\frac{(\mathbf{x}-\mathbf{y},\mathbf{s}_x)}{|\mathbf{x}-\mathbf{y}|^2}\,ds_y - \frac{1}{2\pi}\int_{C_n}[\nu(\mathbf{y})-\nu(\mathbf{x})]\frac{(\mathbf{x}-\mathbf{y},\mathbf{s}_x)}{|\mathbf{x}-\mathbf{y}|^2}\,ds_y \\ &\quad -\nu(\mathbf{x})\int_{C_n}\frac{(\mathbf{x}-\mathbf{y},\mathbf{s}_x)}{|\mathbf{x}-\mathbf{y}|^2}\,ds_y\,, \quad \mathbf{x}\in C_n\,, \quad 1\leqslant n\leqslant N\,.\end{aligned} \tag{4.43}$$

Analogous to (4.28) it can be shown that

$$\frac{(\mathbf{x}-\mathbf{y},\mathbf{s}_x)}{|\mathbf{x}-\mathbf{y}|^2} = \frac{\sin(\theta-\phi)}{2(1-\cos(\theta-\phi))}\,, \tag{4.44}$$

which is an odd function of ϕ around $(\theta+\pi)$, and, hence, the last integral in the right-hand side of (4.43) is equal to zero. Thus we obtain from (4.42)–(4.43)

$$\begin{aligned}\frac{\partial V}{\partial N} &= \frac{1}{2\pi}\frac{\partial}{\partial N}\sum_{m=1}^{N} c_m \log|\mathbf{x}-\mathbf{x}_m| - \frac{1}{2\pi}\int_{C\setminus C_n}\nu(y)\frac{(\mathbf{x}-\mathbf{y},\mathbf{s}_x)}{|x-y|^2}\,ds_y \\ &\quad -\int_{C_n}[\nu(\mathbf{y})-\nu(\mathbf{x})]\frac{(\mathbf{x}-\mathbf{y},\mathbf{s}_x)}{|\mathbf{x}-\mathbf{y}|^2}\,ds_y\,, \quad \mathbf{x}\in C_n\,.\end{aligned} \tag{4.45}$$

When $\nu(\mathbf{x})$ is known, i.e. solved from (4.41), $\partial V/\partial N$ can be calculated from (4.45).

Before proceeding with the explicit numerical calculations that will be presented in the next section, we recapitulate here the main steps in the calculation of A_{mn}. This procedure is built up in three parts, namely (for $V=\psi$, $V=u_k$, and $V=\psi_m$, respectively)

Part 1: $V=\psi$.

i) Calculate $\mu(\mathbf{x})$ from (4.16) with $f(\mathbf{x})$ according to (4.8.1).
ii) Calculate $\mu^l(\mathbf{x})$ from (4.20) (note that this relation and, hence, also μ^l, is identical for each V).
iii) Determine $\mathbf{a}$ and B from their definitions (i.e. (4.30), (4.34)) and solve (4.35) for $(\mathbf{c}^T, c_0)^T$; this also yields $\psi(\infty)=c_0$ (see (4.24)).
iv) Calculate $\nu(\mathbf{x})$ from (4.41) together with (4.39)[3].
v) Find $\partial\psi/\partial N$ from (4.45).

Part 2: $V=u_k$, $1\leqslant k\leqslant N$.

i)–v) Analogous to Part 1, only with $f(\mathbf{x})$ from (4.8.3), whereas in iii) and v) $u_k(\infty)$ and $\partial u_k/\partial N$, respectively, are obtained
vi) Calculate α_k from (4.9)[1] and (4.11)[1].
vii) Find $\partial\omega/\partial N$ from (4.6)[1].

Part 3: $V = \psi_m$, $1 \leq m \leq N$.

i) Use the result from Part 2 vii) to obtain $f(\mathbf{x})$ from (4.8.2), and calculate $\mu(\mathbf{x})$ from (4.16).
ii) Take $\mu^l(\mathbf{x})$ from Part 1 ii)
iii) Solve $(\mathbf{c}^T, c_0)^T$ analogous to Part 1 iii), which also gives $\psi_m(\infty) = c_0$.
iv) Calculate $\nu(\mathbf{x})$ from (4.41) and (4.39)[3].
v) Find $\partial\psi_m/\partial N$ from (4.45).
vi) Calculate β_{mn} from (4.9)[2] and (4.11)[2].
vii) Find $\partial\omega_m/\partial N$ from (4.6)[2].

The final step is then:

Use the results of Part 2 vii) and Part 3 vii) for the calculation of A_{mn} $(1 \leq m, n \leq N)$ from (4.12).

5. Numerical evaluation and results

In the preceding section we described a procedure for the solutions of the exterior Dirichlet problem in two dimensions, especially directed towards the calculation of the normal derivatives of the magnetic potentials on the boundaries. In this procedure the Dirichlet problem was reformulated in terms of integral equations. In our numerical program all occurring integral equations are approximated by systems of linear algebraic equations by means of discretization. For the approximations of the integrals and of the tangential derivative of V_3 we use trapezoidal rules and central differences, respectively. The integrand of the last term on the right-hand side of (4.45) in case $\mathbf{y} \to \mathbf{x}$ equals $\partial v/\partial s$, and, again, the latter is approximated by a central difference. The discretization is accomplished by dividing the circles $C_1, \ldots, C_N$ in M segments, each with angle $h = 2\pi/M$. The $\mathbf{x}$- and $\mathbf{y}$-coordinates of the associated nodal points are consecutively numbered as

$$\mathbf{x}_{(k-1)M+j} = [2(k-1)a + \cos(j-1)h]\mathbf{e}_1 + [\sin(j-1)h]\mathbf{e}_2 ,$$

for $\mathbf{x} \in C_k$, and (5.1)

$$\mathbf{y}_{(l-1)M+j} = [2(l-1)a + \cos(j-1)h]\mathbf{e}_1 + [\sin(j-1)h]\mathbf{e}_2 ,$$

for $\mathbf{y} \in C_l$, with

$$k, l \in [1, N] , \quad j \in [1, M], \quad \text{and} \quad h = \frac{2\pi}{M} .$$

In our numerical program we follow the calculation scheme recapitulated at the end of Section 4, but we compute the matrix elements A_{mn} for $m < n$ only; the remaining ones follow from the identities

$$\sum_{n=1}^{N} A_{mn} = 0 , \quad \text{and } A_{mn} = A_{nm} , \tag{5.2}$$

(see (3.20–21) and (2.17)). Standard routines, such as the partial pivoting process, are used

for the solution of the obtained linear systems and for the calculation of the eigenvalues and eigenvectors of an $N \times N$-matrix. As a check for the accuracy of our numerical procedure we compare our results for $N = 2$ with those obtained earlier in [2]. Our results for κ/π correspond to the values of Q_s in [2], Table 4. The results for κ/π, obtained for $M = 40$, and for Q_s are listed in Table 1. We conclude that a very close agreement between κ/π and Q_s exists.

For $N = 2$, the first buckling mode (corresponding to the lowest buckling value or largest eigenvalue κ) is found to be

$$\mathbf{v} = \left[\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}\right]^T, \tag{5.3}$$

again in accordance with the results of [2].

Of course, also the eigenvalue $\kappa = 0$ appears, with buckling mode

$$\mathbf{v} = \left[\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right]^T, \tag{5.4}$$

for A is singular. However, this eigenvalue has no practical relevance, because it yields an infinitely high buckling current. The same phenomenon arises for $N > 2$. Therefore, in the sequel the eigenvalue $\kappa = 0$ is left out of consideration.

In Tables 2, 3 and 4 one finds the numerical results for the eigenvalue κ (related to the buckling current according to (3.19)) and the eigenvector (or buckling modes) for $N = 3$, 4 and 5, respectively; here we have used $M = 40$ and $a/R = 3$.

The values for the buckling current I_0, associated with the computed highest values of κ, can be obtained from (3.19). With

$$I_y = \frac{\pi}{4} R^4, \tag{5.5}$$

Table 1. Values of κ/π for $N = 2$ and $M = 40$ and of Q_s (from [2], Table 4) for various values of a/R

a/R	1.5	2	3	4	6	8	10
κ/π	0.2205	0.1678	0.09328	0.05661	0.02653	0.01520	0.009810
Q_s	0.220	0.168	0.0935	0.0568	0.0266	0.0153	0.00985

Table 2. The eigenvalues and buckling modes for $N = 3$ and $a/R = 3$, computed for $M = 40$

κ/π	v_1	v_2	v_3
0.1393	−0.408	0.816	−0.408
0.0724	0.707	0	−0.707

Table 3. The eigenvalues and buckling modes for $N = 4$ and $a/R = 3$, computed for $M = 40$

κ/π	v_1	v_2	v_3	v_4
0.1640	−0.238	0.666	−0.666	0.238
0.1183	0.500	−0.500	−0.500	0.500
0.0592	−0.666	−0.238	0.238	0.666

Table 4. The eigenvalues and buckling modes for $N = 5$ and $a/R = 3$, computed for $M = 40$

κ/π	v_1	v_2	v_3	v_4	v_5
0.1790	−0.144	0.490	−0.692	0.490	0.144
0.1459	0.335	−0.623	0	0.623	−0.335
0.1028	−0.528	0.245	0.566	0.245	−0.528
0.0501	−0.623	−0.335	0	0.335	0.623

Table 5

N	3	4	5
$(I_0)_N/(I_0)_2$	0.818	0.754	0.722

for circular cross-sections, and with

$$\delta = \lambda R = \frac{\pi R}{l}\,, \tag{5.6}$$

for simply supported rods, (3.19) yields

$$I_0 = \frac{1}{\sqrt{\kappa/\pi}}\,\frac{\pi^3 R^3}{l^2}\,\sqrt{\frac{E}{\mu_0}}\,. \tag{5.7}$$

With use of this formula we have compared the results for 3, 4 and 5 rods with the buckling current for a set of 2 rods. The results are listed in Table 5.

6. Discussion

In [2] and [3], as an alternative way, a more technical approach to the solution of buckling problems for (super)conducting structural systems was discussed. The method is based upon a generalization of the law of Biot and Savart (cf. [7], Sect. 2.6). In [2] this method was applied to the problem of two parallel rods. In a straightforward derivation, completely analogous to that of [2], this method can be generalized to systems of more than 2 rods. For instance, for three rods the following equations are obtained

$$\begin{aligned} EI_y v_1^{iv}(z) &= k_1(v_1 - v_2) + \frac{1}{4}\,k_1(v_1 - v_3)\,,\\ EI_y v_2^{iv}(z) &= k_1(2v_2 - v_1 - v_3)\,,\\ EI_y v_3^{iv}(z) &= k_1(v_3 - v_2) + \frac{1}{4}\,k_1(v_3 - v_1)\,, \end{aligned} \tag{6.1}$$

with

$$k_1 = \frac{\mu_0 I_0^2}{8\pi a^2}\,. \tag{6.2}$$

Under the boundary conditions

$$v_i(0) = v_i''(0) = v_i(l) = v_i''(l)\,, \quad i = 1, 2, 3\,, \tag{6.3}$$

the lowest eigenvalue of (6.1) is

$$k_1 = \frac{\pi^4 E I_y}{3l^4}\,, \tag{6.4}$$

associated with the buckling mode

$$v_1(z) = v_3(z) = -\frac{1}{2}\, v_2(z)\,, \qquad v_2(z) = A \sin\left(\frac{\pi z}{l}\right). \tag{6.5}$$

This buckling mode is identical to the first one of Table 2.

From (6.4) with $(6.2)^1$ the following formula for the buckling current is found (with $I_y = \pi R^4/4$)

$$I_0 = \sqrt{\frac{2}{3}}\, \frac{\pi^3 a R^2}{l^2} \sqrt{\frac{E}{\mu_0}}\,. \tag{6.6}$$

Let us compare this result with (5.7). For $a/R = 3$ we obtain from (5.7)

$$I_0 = 2.679\, \frac{\pi^3 R^3}{l^2} \sqrt{\frac{E}{\mu_0}}\,, \tag{6.7}$$

and from (6.5)

$$I_0 = 2.449\, \frac{\pi^3 R^3}{l^2} \sqrt{\frac{E}{\mu_0}}\,. \tag{6.8}$$

We see that the buckling value found by the Biot–Savart method is about 8% lower than the value from the variational method. The same difference was also found in [2] for the set of two rods.

For the system of 5 rods, the Biot–Savart method yields the buckling mode

$$v_2 = v_4 = -0.72 v_3\,, \qquad v_1 = v_5 = 0.22 v_3\,, \tag{6.9}$$

which differs only slightly from the first buckling mode from Table 4, where

$$v_2 = v_4 = -0.708 v_3\,, \qquad v_1 = v_5 = 0.208 v_3\,. \tag{6.10}$$

For the buckling current we obtained

$$I_0 = 0.723\, \frac{\pi^3 a R^2}{l^2} \sqrt{\frac{E}{\mu_0}}\,, \tag{6.11}$$

yielding, for $a/R = 3$,

$$I_0 = 2.168\, \frac{\pi^3 R^3}{l^2} \sqrt{\frac{E}{\mu_0}}\,. \tag{6.12}$$

On the other hand, (5.7) gives for $a/R = 3$

$$I_0 = 2.364 \frac{\pi^3 R^3}{l^2} \sqrt{\frac{E}{\mu_0}}, \tag{6.13}$$

and again a difference of about 8% is observed. Hence, we conclude that this relative difference is independent of the number N.

Finally, we also calculated by the Biot–Savart method the buckling current for an infinite set of parallel rods. The result was that the buckling modes were related to each other by

$$v_{j+1} = -v_j, \quad j = 1, 2, \ldots, \tag{6.14}$$

while the buckling current was found to be

$$I_0 = \frac{\pi^2 a R^2}{2l^2} \sqrt{\frac{E}{\mu_0}}. \tag{6.15}$$

It is striking to note that this value for the infinite set is exactly a factor $(\pi/2)$ lower than the value for the set of two rods, which according to [2], (5.31) is equal to

$$I_0 = \frac{\pi^3 a R^2}{l^2} \sqrt{\frac{E}{\mu_0}}. \tag{6.16}$$

We proceed with the analogous version of Table 5, but now with the results from the Biot–Savart method (see Table 6). We note that the above ratios are independent of the value of a/R. Moreover, the differences in the ratios according to Table 5 and to Table 6 (for $N = 3$ or 5) are negligible. Hence, we may write (the subindices V and BS denote values according to the variational method and the Biot–Savart method, respectively)

$$\left(\frac{I_{0N}}{I_{02}}\right)_V = \left(\frac{I_{0N}}{I_{02}}\right)_{BS} = q_N(N), \tag{6.17}$$

where q_N depends only on N and not on a/R. With the use of [2], (5.17), this relation implies that

$$(I_{0N})_V = \frac{q_N}{\sqrt{Q_s}} \frac{\pi^3 R^3}{l^2} \sqrt{\frac{E}{\mu_0}}. \tag{6.18}$$

If we assume this relation of general validity (i.e. for all values of a/R and N) we can extrapolate the results of Table 5 for $N = 3$ and $N = 5$ to other values of a/R. To this end we use the $1/\sqrt{Q_S}$-values as given in [2], Table 4, for several values of a/R. Furthermore, we can also find a corresponding value for the infinite system. In this way we find for the coefficient i_0 defined by

Table 6. Ratios of the buckling currents for N rods and for 2 rods, calculated by means of the Biot–Savart method

N	3	5	∞
$(I_0)_N/(I_0)_2$	0.816	0.723	0.637

Table 7. Values of the normalized buckling current i_0 found by extrapolation from the Biot–Savart results

a/R	N		
	3	5	∞
4	3.429	3.037	2.674
6	5.005	4.432	3.903
8	6.606	5.850	5.150
10	8.230	7.288	6.417

$$I_0 = i_0 \frac{\pi^3 R^3}{l^2} \sqrt{\frac{E}{\mu_0}}, \tag{6.19}$$

the relation

$$i_0 = \frac{q_N}{\sqrt{Q_S}} = i_0\left(\frac{a}{R}, N\right). \tag{6.20}$$

Values for this normalized buckling current are listed in Table 7.

In conclusion, we state that we have found here a simple algorithm to extrapolate from the Biot–Savart results the buckling values (more exact but also much harder to obtain) as they should be found by the variational method. Due to the striking correspondence between systems of rods and (parallel) rings, as found in [3], it may be expected that this result can be generalized to sytems of $N(N \geq 2)$ rings. This will enable us to apply a combined method (based partially upon a variational approach and partially on Biot–Savart like calculations) to more complex systems such as, for instance, helical or spiral shaped conductors (cf. [8]).

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